

GAUSSIAN PROCESSES
EXERCISE SHEET 8: LINEAR STRUCTURE OF GAUSSIANS II

Exercise 1 (Conditioning Gaussians). Let (\bar{X}, \bar{Y}) be a zero mean Gaussian vector in \mathbb{R}^{n+m} and $\bar{X} \in \mathbb{R}^n$. Show that (\bar{X}, \bar{Y}) has the same law as $(\bar{X}, C_{YX}C_{XX}^{-1}\bar{X} - \bar{Z})$, where \bar{Z} is a zero mean Gaussian with covariance $C_{YY} - C_{YX}C_{XX}^{-1}C_{XY}$.

Exercise 2 (Gaussian random walk revisited). Recall the Gaussian random walk defined on $\{0, 1, \dots, n\}$. Find the conditional distribution of Γ , conditioned on $\Gamma(n) = 0$. More precisely, denoting this walk by $\hat{\Gamma}$,

- find the covariance kernel \hat{C} of $\hat{\Gamma}$,
- guess the inverse of the covariance kernel \hat{C} ,
- using similar steps as in the class, find a Fourier representation of $\hat{\Gamma}$.

Finally, show that the conditioned random walk $\hat{\Gamma}$ has the same law as $\Gamma - \Gamma(n)$. How to think about that?

Exercise 3 (Bayesian linear regression). In the usual linear regression we try to model some data $(x_i, y_i)_{i=1, \dots, n}$ using a linear function $y_i = Ax_i + \epsilon_i$, where A is a parameter to be determined and ϵ_i are i.i.d. Gaussians of some small variance σ^2 . In the Bayesian setting, one takes A itself to be an independent random variable with some a priori distribution – let us say for simplicity that we start from $A_p \sim N(0, 1)$ – that one updates based on the observations.

- Given (x_1, y_1) and thus the condition $y_1 = Ax_1 + \epsilon_1$, what is the conditional law of (A, ϵ_1) ?
- What is the conditional law of A , given the n -tuple $(x_i, y_i)_{i=1 \dots n}$?

Now suppose that $(x_i, y_i)_{i=1 \dots n}$ are i.i.d. examples of the vector $(X, 2X)$ with $X \sim N(0, 1)$. What does the conditional law of A converge to as $n \rightarrow \infty$?

Exercise 4 (Conditioning Gaussians II). Let $X \in \mathbb{R}^n$ be a Gaussian random vector with mean 0 and positive definite covariance matrix C . Consider a linear conditioning $MX = 0$, where M is an $m \times n$ matrix of rank m .

Find the conditional law of X in terms of the covariance matrix C and the matrix M .